

THE METRICAL THEORY OF SIMULTANEOUSLY SMALL LINEAR FORMS.

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ABSTRACT. In this paper we investigate the metrical theory of Diophantine approximation associated with linear forms that are simultaneously small for infinitely many integer vectors; i.e. forms which are close to the origin. A complete Khintchine–Groshev type theorem is established, as well as its Hausdorff measure generalization. The latter implies the complete Hausdorff dimension theory.

1. INTRODUCTION

Let $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a real positive decreasing function with $\psi(r) \rightarrow 0$ as $r \rightarrow \infty$. Such a function will be referred to as an *approximation* function. An $m \times n$ matrix $X = (x_{ij}) \in \mathbb{R}^{mn}$ is said to be ψ -*approximable* if the system of inequalities

$$|q_1 x_{1i} + q_2 x_{2i} + \cdots + q_m x_{mi}| \leq \psi(|\mathbf{q}|) \quad \text{for } (1 \leq i \leq n),$$

is satisfied for infinitely many $\mathbf{q} \in \mathbb{Z}^m \setminus \{0\}$. Here and throughout $|\mathbf{q}|$ will denote the supremum norm of the vector \mathbf{q} . Specifically, $|\mathbf{q}| = \max\{|q_1|, |q_2|, \dots, |q_m|\}$. The system $q_1 x_{1i} + q_2 x_{2i} + \cdots + q_m x_{mi}$ of n linear forms in m variables q_1, q_2, \dots, q_m will be written more concisely as $\mathbf{q}X$, where the matrix X is regarded as a point in \mathbb{R}^{mn} . It is easily verified that ψ -approximability is not affected under translation by integer vectors and we can therefore restrict attention to the unit cube $\mathbb{I}^{mn} := [-\frac{1}{2}, \frac{1}{2}]^{mn}$. The set of ψ -approximable points in \mathbb{I}^{mn} will be denoted by $W_0(m, n; \psi)$;

$$W_0(m, n; \psi) := \{X \in \mathbb{I}^{mn} : |\mathbf{q}X| < \psi(|\mathbf{q}|) \text{ for i.m. } \mathbf{q} \in \mathbb{Z}^m \setminus \{0\}\},$$

where ‘i.m.’ means ‘infinitely many’. In the case when $\psi(r) = r^{-\tau}$ for some $(\tau > 0)$ we shall write $W_0(m, n; \tau)$ instead of $W_0(m, n; \psi)$.

It is worth relating the above to the set of ψ -well approximable matrices as is often studied in classical Diophantine approximation. In such a setting studying the metric structure of the lim sup-set

$$W(m, n; \psi) = \{X \in \mathbb{I}^{mn} : \|\mathbf{q}X\| < \psi(|\mathbf{q}|) \text{ for i.m. } \mathbf{q} \in \mathbb{Z}^m \setminus \{0\}\},$$

where $\|x\|$ denotes the distance of x to the nearest integer vector, is a central problem and the theory is well established, see for example [4] or [1]. Probably the main result in this setting is the Khintchine–Groshev theorem which gives an elegant answer to the question of the size of the $W(m, n; \psi)$. The result links the measure of the set to the convergence or otherwise of a series that depends only on the approximating function and is the template for many results in the field of metric number theory. It is clear then that the set $W_0(m, n; \psi)$ is an analogue

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of $W(m, n; \psi)$ with $|\cdot|$ replacing $\|\cdot\|$. The aim of this paper is to obtain the complete metric theory for the set $W_0(m, n; \psi)$.

It is readily verified that $W_0(1, n; \psi) = \{0\}$ as any $x = (x_1, x_2, \dots, x_n) \in W_0(1, n; \psi)$ must satisfy the inequality $|qx_j| < \psi(q)$ infinitely often. As $\psi(q) \rightarrow 0$ as $q \rightarrow \infty$ this is only possible if $x_j = 0$ for all $j = 1, 2, \dots, n$. Thus when $m = 1$ the set $W_0(1, n; \psi)$ is a singleton and must have both zero measure and dimension. We will therefore assume that $m \geq 2$.

Before giving the main results of this paper we include a brief review of some of the work done previously on the measure theoretic structure of $W_0(m, n; \psi)$.

The first result is due to Dickinson [5].

Theorem (Dickinson). When $\tau > \frac{m}{n} - 1$ and $m \geq 2$

$$\dim(W_0(m, n; \tau)) = (m-1)n + \frac{m}{\tau+1},$$

and when $0 < \tau \leq \frac{m}{n} - 1$,

$$\dim(W_0(m, n; \tau)) = mn.$$

It turns out that Dickinson's original result is false when $m \leq n$. The correct statement is given in Corollary 5 which is a consequence of Theorem 2 proved below. To the best of our knowledge the only other result is due to Kemble [12] who established a Khintchine–Groshev type theorem for $W_0(m, 1; \psi)$ under various conditions on the approximating function. We shall remove these conditions and prove the precise analogue of the Khintchine–Groshev theorem for $W_0(m, n; \psi)$. Finally, it is worth mentioning that the set is not only of number theoretic interest but appears naturally in operator theory, see [6] for further details.

Notation. To simplify notation the symbols \ll and \gg will be used to indicate an inequality with an unspecified positive multiplicative constant. If $a \ll b$ and $a \gg b$ we write $a \asymp b$, and say that the quantities a and b are comparable. For a set A , $|A|_k$ will be taken to mean the k -dimensional Lebesgue measure of the set A .

2. STATEMENT OF MAIN RESULTS

The results of this paper depend crucially on the choice of m and n . We shall see that when $m > n$, the metric theory is 'independent' and for this particular case Dickinson's dimension result is correct. When $m \leq n$ the measure results are dependent on the independent case. Dickinson's result for this particular case is incorrect and we provide the correct result.

In the following \mathcal{H}^f denotes f -dimensional Hausdorff measure which will be defined fully in §3.1. Given an approximating function ψ let $\Psi(r) := \frac{\psi(r)}{r}$.

Theorem 1. *Let $m > n$ and ψ be an approximating function. Let f be a dimension function such that $r^{-mn}f(r)$ is monotonic and $r^{-(m-1)n}f(r)$ is increasing. Then*

$$\mathcal{H}^f(W_0(m, n; \psi)) = \begin{cases} 0 & \text{if } \sum_{r=1}^{\infty} f(\Psi(r))\Psi(r)^{-(m-1)n}r^{m-1} < \infty, \\ \mathcal{H}^f(\mathbb{I}^{mn}) & \text{if } \sum_{r=1}^{\infty} f(\Psi(r))\Psi(r)^{-(m-1)n}r^{m-1} = \infty. \end{cases}$$

The requirement that $r^{-mn}f(r)$ be monotonic is a natural and not particularly restrictive condition. Note that if the dimension function f is such that $r^{-mn}f(r) \rightarrow \infty$ as $r \rightarrow 0$ then $\mathcal{H}^f(\mathbb{I}^{mn}) = \infty$ and Theorem 1 is the analogue of the classical result of Jarník (see [11]).

Theorem 1 implies analogues of both the Lebesgue and Hausdorff measure results familiar from classical Diophantine approximation. In the case when $f(r) := r^{mn}$ the Hausdorff

measure \mathcal{H}^f is simply standard Lebesgue measure supported on \mathbb{I}^{mn} and the result is the natural analogue of the Khintchine–Groshev theorem for $W_0(m, n; \psi)$.

Corollary 1. *Let $m > n$ and ψ be an approximating function, then*

$$|W_0(m, n; \psi)|_{mn} = \begin{cases} 0 & \text{if } \sum_{r=1}^{\infty} \psi(r)^n r^{m-n-1} < \infty, \\ 1 & \text{if } \sum_{r=1}^{\infty} \psi(r)^n r^{m-n-1} = \infty. \end{cases}$$

If we now set $f : r \rightarrow r^s (s > 0)$ then Theorem 1 reduces to the following s -dimensional Hausdorff measure statement which is more discriminating than the Hausdorff dimension result of Dickinson.

Corollary 2. *Let $m > n$ and ψ be an approximating function. Let s be such that $(m-1)n < s \leq mn$. Then,*

$$\mathcal{H}^s(W_0(m, n; \psi)) = \begin{cases} 0 & \text{if } \sum_{r=1}^{\infty} \Psi(r)^{s-(m-1)n} r^{m-1} < \infty, \\ \mathcal{H}^s(\mathbb{I}^{mn}) & \text{if } \sum_{r=1}^{\infty} \Psi(r)^{s-(m-1)n} r^{m-1} = \infty. \end{cases}$$

Under the conditions of Corollary 2 it follows from the definition of Hausdorff dimension that

$$\dim(W_0(m, n; \psi)) = \inf \left\{ s : \sum_{r=1}^{\infty} \Psi(r)^{s-(m-1)n} r^{m-1} < \infty \right\},$$

and in particular the following dimension result for $W_0(m, n; \tau)$ holds.

Corollary 3. *Let $m > n$ and $\tau > \frac{m}{n} - 1$ then*

$$\dim(W_0(m, n; \tau)) = (m-1)n + \frac{m}{\tau+1}.$$

Theorem 1 establishes the metric theory for $W_0(m, n; \psi)$ when $m > n$. For the cases when $m \leq n$ the statement of Theorem 1 changes somewhat. The sum which determines the f -measure remains the same but the conditions on the dimension functions are different. This is due to the fact that set $W(m, n; \psi)$ can be shown to lie in a manifold $\Gamma \subset \mathbb{R}^{mn}$ of dimension $(m-1)(n+1)$, a fact we prove later in §5. In light of this remark, an upper bound for $\dim W_0(m, n; \psi)$ follows immediately. More specifically,

$$\dim W_0(m, n; \psi) \leq (m-1)(n+1).$$

Theorem 2. *Let $m \leq n$ and ψ be an approximating function. Let f and g be dimension functions with $g(r) = r^{-(n-m+1)(m-1)} f(r)$. Assume that $r^{-(m-1)(n+1)} f(r)$ is monotonic and $r^{-(m-1)n} f(r)$ increasing. Then $\mathcal{H}^f(W_0(m, n; \psi)) = 0$ if*

$$\sum_{r=1}^{\infty} f(\Psi(r)) \Psi(r)^{-(m-1)n} r^{m-1} < \infty.$$

If

$$\sum_{r=1}^{\infty} f(\Psi(r)) \Psi(r)^{-(m-1)n} r^{m-1} = \infty,$$

then

$$\mathcal{H}^f(W_0(m, n; \psi)) = \begin{cases} \infty & \text{if } r^{-(m-1)(n+1)} f(r) \rightarrow \infty \text{ as } r \rightarrow 0, \\ \mathcal{H}^f(\Gamma) & \text{if } r^{-(m-1)(n+1)} f(r) \rightarrow C \text{ as } r \rightarrow 0, \end{cases}$$

where $C > 0$ is some fixed constant.

It is worth noting that for dimension functions f such that $r^{-(m-1)(n+1)} f(r) \rightarrow C > 0$ as $r \rightarrow 0$ the measure \mathcal{H}^f is comparable to standard $(m-1)(n+1)$ -dimensional Lebesgue measure and in the case when $f(r) = r^{(m-1)(n+1)}$, we obtain the following analogue of the Khintchine-Groshev theorem.

Corollary 4. *Let $m \leq n$ and ψ be an approximating function and assume that the conditions of Theorem 2 hold for the dimension function $f(r) := r^{(m-1)(n+1)}$. Then*

$$\mu(W_0(m, n; \psi)) = \begin{cases} 0 & \text{if } \sum_{r=1}^{\infty} \psi(r)^{m-1} < \infty, \\ 1 & \text{if } \sum_{r=1}^{\infty} \psi(r)^{m-1} = \infty, \end{cases}$$

where μ is the normalised measure on the manifold Γ .

As above, if we set $f(r) = r^s$ we obtain the $m \leq n$ analogue of Corollary 2.

Corollary 5. *Let $m \leq n$ and ψ be an approximating function. Let s be such that $(m-1)n < s \leq (m-1)(n+1)$ and let $g : r \rightarrow r^{s-(n-(m-1))(m-1)}$ be a dimension function. Then,*

$$\mathcal{H}^s(W_0(m, n; \psi)) = \begin{cases} 0 & \text{if } \sum_{r=1}^{\infty} \Psi(r)^{s-(m-1)n} r^{m-1} < \infty, \\ \mathcal{H}^s(\Gamma) & \text{if } \sum_{r=1}^{\infty} \Psi(r)^{s-(m-1)n} r^{m-1} = \infty. \end{cases}$$

Further, under the same conditions as Corollary 5, but with the approximation function $\psi(x) = x^{-\tau}$, we have the following result:

Corollary 6. *Let $m \leq n$ and $\tau > \frac{m}{m-1} - 1$. Then*

$$\dim(W_0(m, n; \tau)) = (m-1)n + \frac{m}{\tau+1},$$

and when $0 < \tau \leq \frac{m}{m-1} - 1$,

$$\dim(W_0(m, n; \tau)) = (m-1)(n+1).$$

The paper is organized as follows. In Section 3, we give the definitions of Hausdorff measure and ubiquity, which is the main tool for proving Theorem 1, in a manner appropriate to the setting of this paper. Section 3 also includes the statement of the ‘Slicing’ lemma (Lemma 1) which is used to prove Theorem 2. The paper continues with the proof of Theorem 1 in § 4. As is common when proving such ‘zero-full’ results the proof is split into two parts; the convergence case and the divergence case. We conclude the paper with the proof of Theorem 2.

3. BASIC DEFINITIONS AND AUXILIARY RESULTS

In this section we give definitions of some fundamental concepts along with some auxiliary results which will be needed in the proofs of Theorems 1 and 2.

3.1. Hausdorff Measure and Dimension. Below we give a brief introduction to Hausdorff f -measure and dimension. For further details see [9].

Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be an increasing continuous function such that $f(r) \rightarrow 0$ as $r \rightarrow 0$. Such a function f is referred to as a *dimension function*. We are now in a position to define the Hausdorff f -measure $\mathcal{H}^f(X)$ of a set $X \subset \mathbb{R}^n$.

Let B be a (Euclidean) ball in \mathbb{R}^n . That is a set of the form

$$B = \{x \in \mathbb{R}^n : |x - c|_2 < \delta\}$$

for some $c \in \mathbb{R}^n$ and some $\delta > 0$. The *diameter* $\text{diam}(B)$ of B is

$$\text{diam}(B) := \sup\{|x - y|_2 : x, y \in B\}.$$

Now for any $\rho > 0$ a countable collection $\{B_i\}$ of balls in \mathbb{R}^n with diameters $\text{diam}(B_i) \leq \rho$ such that $X \subset \bigcup_i B_i$ is called a ρ -cover for X . Define

$$\mathcal{H}_\rho^f(X) = \inf \left\{ \sum_i f(\text{diam}(B_i)) : \{B_i\} \text{ is a } \rho\text{-cover for } X \right\},$$

where the infimum is taken over all possible ρ -covers of X . The Hausdorff f -measure of X is defined to be

$$\mathcal{H}^f(X) = \lim_{\rho \rightarrow 0} \mathcal{H}_\rho^f(X).$$

In the particular case when $f(r) := r^s$ ($s > 0$), we write $\mathcal{H}^s(X)$ for \mathcal{H}^f and the measure is referred to as s -dimensional Hausdorff measure. The Hausdorff dimension of a set X is denoted by $\dim(X)$ and is defined as follows,

$$\dim(X) := \inf\{s \in \mathbb{R}^+ : \mathcal{H}^s(X) = 0\} = \sup\{s \in \mathbb{R}^+ : \mathcal{H}^s(X) = \infty\}.$$

Note that the value of $\dim(X)$ is unique. At the critical exponent $s = \dim X$ the quantity $\mathcal{H}^s(X)$ is either zero, infinite or strictly positive and finite. In the latter case; i.e. when

$$0 < \mathcal{H}^s(X) < \infty,$$

the set X is said to be an s -set.

3.2. Ubiquitous Systems. To make this article as self contained as possible we describe the main tool used in proving the divergence part of Theorem 1, the idea of a locally ubiquitous system. The set-up presented below is simplified for the current problem. The general framework is much more abstract and full details can be found in [4] and [1].

Let $\mathfrak{R} = \{R_{\mathbf{q}} : \mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}\}$ be the family of subsets $R_{\mathbf{q}} := \{X \in \mathbb{I}^{mn} : \mathbf{q}X = \mathbf{0}\}$. The sets $R_{\mathbf{q}}$ will be referred to as *resonant sets*. Let the function $\beta : \mathbb{Z}^m \setminus \{\mathbf{0}\} \rightarrow \mathbb{R}^+ : \mathbf{q} \rightarrow |\mathbf{q}|$ attach a weight to the resonant set $R_{\mathbf{q}}$. Now, given an approximating function ψ and $R_{\mathbf{q}}$, let

$$\Delta(R_{\mathbf{q}}, \Psi(|\mathbf{q}|)) := \left\{ X \in \mathbb{I}^{mn} : \text{dist}(X, R_{\mathbf{q}}) \leq \frac{\psi(|\mathbf{q}|)}{|\mathbf{q}|} \right\}$$

where $\text{dist}(X, R_{\mathbf{q}}) := \inf\{|X - Y| : Y \in R_{\mathbf{q}}\}$. Thus $\Delta(R_{\mathbf{q}}, \Psi(|\mathbf{q}|))$ is a Ψ -neighbourhood of $R_{\mathbf{q}}$. Notice that in the case when the resonant sets are points the sets $\Delta(R_{\mathbf{q}}, \Psi(|\mathbf{q}|))$ are simply balls centred at resonant points.

Let

$$\Lambda(m, n; \psi) = \{X \in \mathbb{I}^{mn} : X \in \Delta(R_{\mathbf{q}}, \Psi(|\mathbf{q}|)) \text{ for i.m. } \mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}\}.$$

The set $\Lambda(m, n; \psi)$ is a ‘limsup’ set. It consists entirely of points in \mathbb{I}^{mn} which lie in infinitely many of the sets $\Delta(R_q, \Psi(|\mathbf{q}|))$. This is apparent if we restate $\Lambda(m, n; \psi)$ in a manner which emphasises its limsup structure.

Fix $k > 1$ and for any $t \in \mathbb{N}$, define

$$\Delta(\psi, t) := \bigcup_{k^{t-1} \leq |\mathbf{q}| \leq k^t} \Delta(R_{\mathbf{q}}, \Psi(|\mathbf{q}|)). \quad (1)$$

It follows that

$$\Lambda(m, n; \psi) = \limsup_{t \rightarrow \infty} \Delta(\psi, t) = \bigcap_{N=1}^{\infty} \bigcup_{t=N}^{\infty} \Delta(\psi, t). \quad (2)$$

The key point by which ubiquity will be utilised is in the fact that the sets $W_0(m, n; \psi)$ and $\Lambda(m, n; \psi)$ actually coincide.

We now move onto the formal definition of a locally ubiquitous system. As stated above the definition given below is in a much simplified form suitable to the problem at hand. In the more abstract setting given in [1] there are specific conditions on both the measure on the ambient space and its interaction with neighbourhoods of the resonant set which must be shown to hold. These conditions are not stated below as they hold trivially for Lebesgue measure, the measure on our ambient space \mathbb{I}^{mn} , and stating the conditions would complicate the discussion somewhat. Never the less, the reader should be aware that in the more abstract notion of ubiquity these extra conditions exist and need to be established.

Let $\rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a function with $\rho(r) \rightarrow 0$ as $r \rightarrow \infty$ and let

$$\Delta(\rho, t) := \bigcup_{\mathbf{q} \in J(t)} \Delta(R_{\mathbf{q}}, \rho(k^t))$$

where $J(t)$ is defined to be the set

$$J(t) := \{\mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\} : |\mathbf{q}| \leq k^t\}$$

for a fixed constant $k > 1$.

Definition 1. Let $B := B(X, r)$ be an arbitrary ball with centre $X \in \mathbb{I}^{mn}$ and $r \leq r_o$. Suppose there exists a function ρ and an absolute constant $\kappa > 0$ such that

$$|B \cap \Delta(\rho, t)|_{mn} \geq \kappa |B|_{mn} \text{ for } t \geq t_o(B).$$

Then the pair (\mathfrak{R}, β) is said to be a *locally ubiquitous* system relative to (ρ, k) .

Loosely speaking the definition of local ubiquity says that the set $\Delta(\rho, t)$ locally approximates the underlying space \mathbb{I}^{mn} in terms of the Lebesgue measure. The function ρ , will be referred to as the *ubiquity function*. The actual values of the constants κ and k in the above definition are irrelevant, it is their existence that is important. In practice the local ubiquity of a system can be established using standard arguments concerning the distribution of the resonant sets in \mathbb{I}^{mn} , from which the function ρ arises naturally.

Clearly if $|\Delta(\rho, t)|_{mn} \rightarrow 1$ as $t \rightarrow \infty$ then (\mathfrak{R}, β) is locally-ubiquitous. To see this let B be any ball and assume without loss of generality that $|B|_{mn} = \epsilon > 0$. Then for t sufficiently large,

$$|\Delta(\rho, t)|_{mn} > 1 - \epsilon/2.$$

Hence $|B \cap \Delta(\rho, t)|_{mn} \geq \epsilon/2$ as required.

Given a positive real number $k > 1$ a function f will be said to be k -regular if there exists a positive constant $\lambda < 1$ such that for t sufficiently large

$$f(k^{t+1}) \leq \lambda f(k^t).$$

Finally, we set $\gamma = \dim(R_{\mathbf{q}})$, the common (Euclidean) dimension of the resonant sets $R_{\mathbf{q}}$. The following theorem is a simplified version of Theorem 1 from [4].

Theorem 3 (BV). *Suppose that (\mathfrak{R}, β) is locally ubiquitous relative to (ρ, k) and ψ is an approximation function. Let f be a dimension function such that $r^{-\delta} f(r)$ is monotonic. Furthermore suppose that $r^{-\gamma} f(r)$ is increasing and ρ is k -regular. Then*

$$\mathcal{H}^f(W_0(m, n; \psi)) = \mathcal{H}^f(\mathbb{I}^{mn}) \quad \text{if} \quad \sum_{n=1}^{\infty} \frac{f(\Psi(k^t)) \Psi(k^t)^{-\gamma}}{\rho(k^t)^{\delta-\gamma}} = \infty. \quad (3)$$

3.3. Slicing. We now state a result which is the crucial key ingredient in the proof of Theorem 2. The result was used in [3] to prove the Hausdorff measure version of the W. M. Schmidt's inhomogeneous linear forms theorem in metric number theory. The authors refer to the technique as "slicing". We will merely state the result. For a more detailed discussion and proof see [3] or [14]. However, before we do state the theorem it is necessary to introduce a little notation.

Suppose that V is a linear subspace of \mathbb{I}^k , V^\perp will be used to denote the linear subspace of \mathbb{I}^k orthogonal to V . Further $V + a := \{v + a : v \in V\}$ for $a \in V^\perp$.

Lemma 1. Let $l, k \in \mathbb{N}$ be such that $l \leq k$ and let f and $g : r \rightarrow r^{-l} f(r)$ be dimension functions. Let $B \subset \mathbb{I}^k$ be a Borel set and let V be a $(k - l)$ -dimensional linear subspace of \mathbb{I}^k . If for a subset S of V^\perp of positive \mathcal{H}^l measure

$$\mathcal{H}^g(B \cap (V + b)) = \infty \quad \forall b \in S,$$

then $\mathcal{H}^f(B) = \infty$.

We are now in a position to begin the proofs of Theorems 1 and 2.

4. THE PROOF OF THEOREM 1

As stated above, the proof of Theorem 1 is split into two parts; the convergence case and the divergence case. We begin with the convergence case as this is more straightforward than the divergence case.

4.1. The Convergence Case. Recall that in the statement of Theorem 1 we assumed that $m > n$ and we imposed some conditions on the dimension function f . As it turns out these conditions are not needed in the convergence case and we can state and prove a much cleaner result which has the added benefit of also implying the convergence case of Theorem 2.

Theorem 4. *Let ψ be an approximating function and let f be a dimension function. If*

$$\sum_{r=1}^{\infty} f(\Psi(r)) \Psi(r)^{-(m-1)n} r^{m-1} < \infty,$$

then

$$\mathcal{H}^f(W_0(m, n; \psi)) = 0.$$

Obviously Theorem 4 implies the convergence cases of Theorems 1 and 2.

Proof. To prove Theorem 4 we make use of the natural cover of $W_0(m, n; \psi)$ given by Equations (1) and (2). It follows almost immediately that for each $N \in \mathbb{N}$ the family

$$\left\{ \bigcup_{R_{\mathbf{q}}: |\mathbf{q}|=r} \Delta(R_{\mathbf{q}}, \Psi(|\mathbf{q}|)) : r = N, N+1, \dots \right\}$$

is a cover for the set $W_0(m, n; \psi)$. That is

$$W_0(m, n; \psi) \subset \bigcup_{r>N} \bigcup_{|\mathbf{q}|=r} \Delta(R_{\mathbf{q}}, \Psi(|\mathbf{q}|))$$

for any $N \in \mathbb{N}$.

Now, for each resonant set $R_{\mathbf{q}}$ let $\Delta(q)$ be a collection of mn -dimensional closed hypercubes C with disjoint interiors and side length $\Psi(|\mathbf{q}|)$ such that

$$C \cap \bigcup_{|\mathbf{q}|=r} \Delta(R_{\mathbf{q}}, \Psi(|\mathbf{q}|)) \neq \emptyset$$

and

$$\Delta(R_{\mathbf{q}}, \Psi(|\mathbf{q}|)) \subset \bigcup_{C \in \Delta(q)} C.$$

Then

$$\#\Delta(q) \ll (\Psi(|\mathbf{q}|))^{-(m-1)n}.$$

where $\#$ denotes cardinality.

Note that

$$W_0(m, n; \psi) \subset \bigcup_{r>N} \bigcup_{|\mathbf{q}|=r} \Delta(R_{\mathbf{q}}, \Psi(|\mathbf{q}|)) \subset \bigcup_{r>N} \bigcup_{\Delta(q): |\mathbf{q}|=r} \bigcup_{C \in \Delta(q)} C.$$

It follows on setting $\rho(N) = \psi(N)$ that

$$\begin{aligned} \mathcal{H}_\rho^f(W_0(m, n; \psi)) &\leq \sum_{r>N} \sum_{\Delta(q): |\mathbf{q}|=r} \sum_{C \in \Delta(q)} f(\Psi(|\mathbf{q}|)) \\ &\ll \sum_{r>N} r^{m-1} f(\Psi(r)) \Psi(r)^{-(m-1)n} \rightarrow 0 \quad \text{as } \rho \rightarrow 0, \end{aligned}$$

and thus from the definition of \mathcal{H}^f -measure that $\mathcal{H}^f(W_0(m, n; \psi)) = 0$, as required. \square

4.2. The Divergence Case. When $m > n$, the divergence part of Theorem 1 relies on the notion of ubiquity and primarily Theorem 3. To use ubiquity we must show that (\mathfrak{R}, β) is locally-ubiquitous with respect to (ρ, k) for a suitable ubiquity function ρ . For the sake of simplicity we fix $k = 2$.

To establish ubiquity we need two technical lemmas. The first of which is due to Dickinson [5] and is an analogue of Dirichlet's theorem. The second is a slight modification again of a result of Dickinson from the same paper. The key difference being the introduction of a function ω instead of \log . We prove only the second result here and the reader is referred to the previously mentioned paper for the proof of Lemma 2.

Lemma 2. For each $X \in \mathbb{I}^{mn}$, there exists a non-zero integer vector \mathbf{q} in \mathbb{Z}^m with $|\mathbf{q}| \leq 2^t$ ($t \in \mathbb{N}$) such that

$$|\mathbf{q}X| < m(2^t)^{-\frac{m}{n}+1}.$$

Lemma 3. Let ω be a positive real increasing function such that $\frac{1}{\omega(t)} \rightarrow 0$ as $t \rightarrow \infty$ and such that for any $C > 1$ and t sufficiently large $\omega(2t) < C\omega(t)$. The family (\mathfrak{R}, β) is locally ubiquitous with respect to the function $\rho : \mathbb{N} \rightarrow \mathbb{R}^+$ where $\rho(t) = m(2^t)^{-\frac{m}{n}}\omega(t)$.

Proof. Throughout this proof \mathbf{q} will refer to those integer vectors which satisfy the conclusion of Lemma 2. Note that a simple calculation will establish the fact that ρ is 2-regular for t sufficiently large. Define now the set $E(t)$ where

$$E(t) = \{X \in \mathbb{I}^{mn} : |\mathbf{q}| < \frac{2^t}{\omega(t)}\}$$

and

$$\Delta(t) = \{X \in \mathbb{I}^{mn} : |X - \partial\mathbb{I}^{mn}| \geq 2^{-t}\} \setminus E(t),$$

$\partial\mathbb{I}^{mn}$ denotes the boundary of the set \mathbb{I}^{mn} .

Then

$$E(t) \subseteq \bigcup_{1 \leq r \leq \frac{2^t}{\omega(t)}} \bigcup_{|\mathbf{q}|=r} \left\{ X \in \mathbb{I}^{mn} : |\mathbf{q}X| < m(2^t)^{-\frac{m}{n}+1} \right\}.$$

Therefore

$$\begin{aligned} |E(t)|_{mn} &\leq \sum_{1 \leq r \leq \frac{2^t}{\omega(t)}} \sum_{|\mathbf{q}|=r} \frac{m^n (2^t)^{-m+n}}{|\mathbf{q}|_2^n} \\ &\ll (2^t)^{-m+n} \sum_{1 \leq r \leq \frac{2^t}{\omega(t)}} r^{m-n-1} \\ &\ll (2^t)^{-m+n} \frac{2^t}{\omega(t)} \left(\frac{2^t}{\omega(t)} \right)^{m-n-1} \\ &= (\omega(t))^{-m+n}. \end{aligned}$$

Therefore, since $m > n$, $\lim_{t \rightarrow \infty} |E(t)|_{mn} \rightarrow 0$ and $\lim_{t \rightarrow \infty} |\mathbb{I}^{mn} \setminus \Delta(t)|_{mn} \rightarrow 0$. Now to show that $|(\Delta(\rho, t))|_{mn} \rightarrow 1$ as $t \rightarrow \infty$, it would be enough to show that $\Delta(t) \subseteq \Delta(\rho, t)$. For this let $X \in \Delta(t) \Rightarrow X \notin E(t)$ and let $\tilde{\mathbf{q}}$ be from lemma 2,

$$\frac{2^t}{\omega(t)} \leq |\tilde{\mathbf{q}}| \leq 2^t.$$

By definition $|\tilde{\mathbf{q}}| = |\tilde{q}_i|$ for some $1 \leq i \leq m$. Let $\delta_j = \frac{-\tilde{\mathbf{q}} \cdot \mathbf{x}^{(j)}}{|\tilde{q}_i|}, j = 1, 2, \dots, n$ so that $\tilde{\mathbf{q}} \cdot (\mathbf{x}^{(j)} + \delta_j \mathbf{e}^{(i)}) = 0$, where $\mathbf{e}^{(i)}$ denotes the i 'th basis vector. Also

$$|\delta_j| = \left| \frac{-\tilde{\mathbf{q}} \cdot \mathbf{x}^{(j)}}{|\tilde{q}_i|} \right| \leq m(2^t)^{-\frac{m}{n}} \omega(t).$$

Therefore $U = (\mathbf{x}^{(j)} + \delta_j \mathbf{e}^{(i)}) = (\mathbf{x}^{(1)} + \delta_1 \mathbf{e}^{(i)}, \mathbf{x}^{(2)} + \delta_2 \mathbf{e}^{(i)} \dots, \mathbf{x}^{(n)} + \delta_n \mathbf{e}^{(i)})$ is a point in $R_{\tilde{\mathbf{q}}}$ and $|X - U| \leq m(2^t)^{-\frac{m}{n}} \omega(t) = \rho(t)$. Hence $X \in \Delta(\rho, t) = \bigcup_{2^{t-1} < |\mathbf{q}| \leq 2^t} \Delta(R_{\mathbf{q}}, \rho(t))$ so that

$$|(\Delta(\rho, t))|_{mn} \rightarrow 1 \text{ as } t \rightarrow \infty.$$

□

We are now almost in position to apply Theorem 3. To this end consider the sum

$$\sum_{t=1}^{\infty} f(\Psi(k^t)) \left(\frac{\Psi(k^t)}{\rho(k^t)} \right)^{\delta-\gamma} \Psi(k^t)^{-\delta},$$

which is comparable to

$$\sum_{t=1}^{\infty} f(\Psi(2^t)) \Psi(2^t)^{-(m-1)n} (2^t)^m \omega(t)^{-n}.$$

Assuming that $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a monotonic function, $\alpha, \beta \in \mathbb{R}$ and $k > 1$. Let f be a dimension function. It is straightforward to show that the convergence or divergence of the sums

$$\sum_{t=1}^{\infty} k^{t\alpha} f(\psi(k^t)) \psi(k^t)^\beta \quad \text{and} \quad \sum_{r=1}^{\infty} r^{\alpha-1} f(\psi(r)) \psi(r)^\beta$$

coincide. By virtue of this fact, the sum in Equation (4.2) is the same as

$$\sum_{r=1}^{\infty} f(\Psi(r)) \Psi(r)^{-(m-1)n} r^{m-1} \omega(r)^{-n}. \quad (4)$$

To obtain the precise statement of the Theorem 1 we need to remove the ω factor from the above. To do this we choose ω in such a way that the sum

$$\sum_{r=1}^{\infty} f(\Psi(r)) \Psi(r)^{-(m-1)n} r^{m-1} \omega(r)^{-n}$$

will converge (*respec.* diverge) if and only if the sum

$$\sum_{r=1}^{\infty} f(\Psi(r)) \Psi(r)^{-(m-1)n} r^{m-1} \quad (5)$$

converges (*respec.* diverges). This is always possible. Firstly, note that if the sum in Equation (4) diverges then so does the sum in Equation (5). On the other hand and if the sum in Equation (5) diverges, then we can find a strictly increasing sequence of positive integers $\{r_i\}_{i \in \mathbb{N}}$ such that

$$\sum_{r_{i-1} \leq r \leq r_i} f(\Psi(r)) \Psi(r)^{-(m-1)n} r^{m-1} > 1$$

and $r_i > 2r_{i-1}$. Now simply define ω be the step function $\omega(r) = i^{\frac{1}{n}}$ for $r_{i-1} \leq r \leq r_i$ and ω satisfies the required properties.

This completes the proof of Theorem 1.

5. PROOF OF THEOREM 2

In view of Theorem 4 we need only prove the divergence part of Theorem 2. The proof will be split into two sub-cases. The first, which we refer to as the “infinite measure” case, is for dimension functions f such that $r^{-(m-1)(n+1)} f(r) \rightarrow \infty$. The second case corresponds to f which satisfy $r^{-(m-1)(n+1)} f(r) \rightarrow C$ for some constant $C > 0$, in which case the measure

is comparable to $(m-1)(n+1)$ -Lebesgue measure and we call this case the “finite measure” case.

We begin the proof of Theorem 2 with the key observation that if $m \leq n$, $W_0(m, n; \psi)$ lies in a manifold of dimension at most $(m-1)(n+1)$.

Consider first the case when $m = n$. Take any $X \in W_0(m, m; \psi)$, then the column vectors of X are linearly dependent. To prove this, assume to the contrary, that the column vectors are linearly independent. Since X is a member of $W_0(m, n; \psi)$ there exists infinitely many \mathbf{q} such that

$$|\mathbf{q}X| < \psi(|\mathbf{q}|).$$

Setting $\mathbf{q}X = \theta$ where $|\theta| < \psi(|\mathbf{q}|)$, as all column vectors are linearly independent X is invertible. Thus

$$|\mathbf{q}| = |\theta X^{-1}|$$

and it follows that

$$1 \leq |\mathbf{q}| = |\mathbf{q}XX^{-1}| \leq C_2(X)\psi(|\mathbf{q}|) \rightarrow 0 \quad \text{as} \quad |\mathbf{q}| \rightarrow \infty.$$

Which is clearly impossible. Therefore the column vectors of X must be linearly dependent and so $\det X = 0$. This in turn implies that X lies on some surface defined by the multinomial equation $\det Y = 0$ where $Y \in \mathbb{I}^{m^2}$. As this equation defines a co-dimension 1 manifold in \mathbb{I}^{m^2} , at most $m^2 - 1$ independent variables are needed to fully specify X .

This above argument is essentially that needed to prove the more general case when $m \leq n$. We prove the result only for the case when $n = m + 1$ as the general case follows with a straightforward modification of the argument given. Given any $X \in W(m, m+1; \psi)$. Thinking of X as an m by $m+1$ matrix as above. Let x be the first column vector, x_2 the $(m+1)$ -th column vector and X' the m by $m-1$ matrix formed by taking the remaining $m-1$ columns of X . Further let X_1 be the $m \times m$ matrix with first column x and remaining columns made up of X' . Similarly let X_2 be the $m \times m$ matrix with first $m-1$ columns the same as X' and final column x_2 . Using the same argument as above we claim that both these sub-matrices of X are in fact non-invertible and so each sub-matrix lies on a co-dimension 1 manifold Γ_i determined by the equation $\det Y_i = 0$ with $i = 1, 2$. Here Y_1 is an $m \times m$ matrix consisting of all but the final m variables of an arbitrary element $Y \in \mathbb{I}^{m(m+1)}$ and Y_2 is similarly defined but the first m variables are now removed. Now X must lie in the intersection of the two manifolds Γ_1 and Γ_2 , say Γ . This is a co-dimension 2 manifold and the result is proved.

With the above observation in mind we begin the proof of Theorem 2 in earnest by defining the set:

$$W_0(m, n; c\psi) := \{X \in \mathbb{I}^{mn} : |\mathbf{q}X| < c\psi(|\mathbf{q}|) \text{ for i.m. } \mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}\}, \quad (6)$$

where $c = \max(\frac{m-1}{2}, 1)$. It is clear then that $W_0(m, n; \psi) \subseteq W_0(m, n; c\psi)$.

Let A be the set of points of the form

$$\left(X^{(1)}, X^{(2)}, \dots, X^{(m-1)}, \sum_{j=1}^{m-1} a_j^{(1)} X^{(j)}, \dots, \sum_{j=1}^{m-1} a_j^{(n-m+1)} X^{(j)} \right),$$

where

$$(X^{(1)}, X^{(2)}, \dots, X^{(m-1)}) \in W_0(m, m-1; \psi)$$

and $a_j^{(i)} \in (-\frac{1}{2}, \frac{1}{2})$ for $1 \leq i \leq (n - m + 1)$. Note that

$$\begin{aligned} \left| \mathbf{q} \cdot \sum_{j=1}^{m-1} a_j^{(i)} X^{(j)} \right| &= \left| \sum_{j=1}^{m-1} a_j^{(i)} \mathbf{q} \cdot X^{(j)} \right| \\ &\leq \sum_{j=1}^{m-1} |a_j^{(i)}| |\mathbf{q} \cdot X^{(j)}| \\ &\leq \left(\sum_{j=1}^{m-1} |a_j^{(i)}| \right) \psi(|\mathbf{q}|) \\ &\leq c\psi(|\mathbf{q}|) \quad 1 \leq i \leq (n - (m - 1)), \end{aligned}$$

and it follows that $A \subseteq W(m, n; c\psi)$.

Now define the function

$$\eta : W_0(m, m - 1, \psi) \times \left(-\frac{1}{2}, \frac{1}{2} \right)^{(n-(m-1))(m-1)} \rightarrow A$$

by

$$\begin{aligned} \eta \left(X^{(1)}, X^{(2)}, \dots, X^{(m-1)}, a_1^1, \dots, a_{m-1}^1, \dots, a_1^{(n-(m-1))}, \dots, a_{m-1}^{(n-(m-1))} \right) = \\ \left(X^{(1)}, X^{(2)}, \dots, X^{(m-1)}, \sum_{j=1}^{m-1} a_j^{(1)} X^{(j)}, \dots, \sum_{j=1}^{m-1} a_j^{(n-m+1)} X^{(j)} \right). \end{aligned}$$

Note that η is surjective and that the vectors X^j , for $j = 1, \dots, m-1$, are linearly independent. This ensures that η is well defined, one-to-one and the Jacobian, $J(\eta)$, of η is of maximal rank. The function η is therefore an embedding and its range is diffeomorphic to A . This in turn implies that η is (locally) bi-Lipschitz.

5.1. The Infinite Measure Case. As mentioned above the proof of Theorem 2 is split into two parts. In this section we concentrate on the infinite measure case which can be deduced from the following lemma.

Lemma 4. Let ψ be an approximating function and let f and $g : r \rightarrow r^{-(n-(m-1))(m-1)} f(r)$ be dimension functions with $r^{-(m-1)(n+1)} f(r) \rightarrow \infty$ as $r \rightarrow 0$. Further, let $r^{-m(m-1)} g(r)$ be monotonic and $r^{-(m-1)^2} g(r)$ be increasing. If

$$\sum_{r=1}^{\infty} f(\Psi(r)) \Psi(r)^{-(m-1)n} r^{m-1} = \infty,$$

then

$$\mathcal{H}^f(A) = \infty.$$

Proof. As η is bi-Lipschitz, we have that

$$\begin{aligned} \mathcal{H}^f(A) &= \mathcal{H}^f \left(\eta \left(W_0(m, m - 1, \psi) \times \mathbb{I}^{(n-(m-1))(m-1)} \right) \right) \\ &\asymp \mathcal{H}^f \left(W_0(m, m - 1, \psi) \times \mathbb{I}^{(n-(m-1))(m-1)} \right). \end{aligned}$$

The proof relies on the slicing technique of Lemma 1. Let $B := W_0(m, m-1; \psi) \times \mathbb{I}^{(n-(m-1))(m-1)} \subseteq \mathbb{I}^{(m-1)(n+1)}$ and V be the space $\mathbb{I}^{m(m-1)} \times \{0\}^{(m-1)(n+1-m)}$. As $W_0(m, m-1; \psi)$ is a limsup set, B is a Borel set. We know that $\dim W_0(m, m-1; \psi) = m(m-1)$ by Theorem 1 and this means that $W_0(m, m-1; \psi)$ is dense in $\mathbb{I}^{m(m-1)}$. Let $S := \{0\}^{m(m-1)} \times \mathbb{I}^{(n+1-m)(m-1)}$. Clearly S is a subset of V^\perp , and further it has positive $\mathcal{H}^{(n-(m-1))(m-1)}$ -measure. Now for each $b \in S$

$$\begin{aligned} \mathcal{H}^g(B \cap (V+b)) &= \mathcal{H}^g\left((W_0(m, m-1; \psi) \times \mathbb{I}^{(n-(m-1))(m-1)}) \cap (V+b)\right) \\ &= \mathcal{H}^g(\widetilde{W}_0(m, m-1; \psi) + b) \\ &= \mathcal{H}^g(\widetilde{W}_0(m, m-1; \psi)), \end{aligned}$$

where $\widetilde{W}_0(m, m-1; \psi) = W(m, m-1; \psi) \times \{0\}^{(n+1-m)(m-1)}$. Thus the g -measure of $\widetilde{W}_0(m, m-1; \psi)$ coincides with the g -measure of $W(m, m-1; \psi)$ and Now applying Theorem 1 with $n = m-1$ implies that $\mathcal{H}^g(B \cap (V+b)) = \infty$ if

$$\sum_{r=1}^{\infty} r^{m-1} g(\Psi(r)) \Psi(r)^{-(m-1)^2} = \infty.$$

Applying Lemma 1, we have $\mathcal{H}^f(A) = \infty$ if

$$\sum_{r=1}^{\infty} r^{m-1} g(\Psi(r)) \Psi(r)^{-(m-1)^2} = \infty$$

and we conclude that $\mathcal{H}^f(A) = \infty$ if

$$\sum_{r=1}^{\infty} f(\Psi(r)) \Psi(r)^{-(m-1)n} r^{m-1} = \infty,$$

as required. \square

We can now complete the proof of Theorem 2. As $A \subseteq W_0(m, n; c\psi)$, $\mathcal{H}^f(A) = \infty$ implies that $\mathcal{H}^f(W_0(m, n; c\psi)) = \infty$ and we need only show that the value of the constant c is irrelevant. Recall that $c \geq 1$. For convenience let $\psi_c(r) := \frac{\psi(r)}{c}$, $\Psi_c(r) := \frac{\Psi(r)}{c}$, $\sum := \sum_{r=1}^{\infty} f(\Psi(r)) \Psi(r)^{-(m-1)n} r^{m-1}$ and $\sum_c := \sum_{r=1}^{\infty} f(\Psi_c(r)) \Psi_c(r)^{-(m-1)n} r^{m-1}$. Since $r^{-(m-1)(n+1)} f(r)$ is decreasing it follows that

$$\infty = \sum \leq c_1 \sum_c$$

where $c_1 = c^{-(m-1)(n+1)}$. Therefore $\sum_c = \infty$ if $\sum = \infty$ and we have $\mathcal{H}^f(W_0(m, n; c\psi_c)) = \infty$ if $\sum_{r=1}^{\infty} f(\Psi(r)) \Psi(r)^{-(m-1)n} r^{m-1} = \infty$. Finally, it follows that $\mathcal{H}^f(W_0(m, n; \psi)) = \infty$ if $\sum_{r=1}^{\infty} f(\Psi(r)) \Psi(r)^{-(m-1)n} r^{m-1} = \infty$, as required.

5.2. Finite measure case. We now come onto the case where $r^{-(m-1)(n+1)} f(r) \rightarrow C$ as $r \rightarrow 0$ and $C > 0$ is finite. In this case \mathcal{H}^f is comparable to $(m-1)(n+1)$ -dimensional Lebesgue measure. Note that in this case the divergence of the sum

$$\sum_{r=1}^{\infty} f(\Psi(r)) \Psi(r)^{-(m-1)n} r^{m-1}$$

is in direct correspondence with that of the sum

$$\sum_{r=1}^{\infty} \psi^{m-1}(r).$$

We begin with the following general lemma, the proof of which we leave to the reader.

Lemma 5. Suppose that $L \subset \mathbb{R}^l$, $M \subset \mathbb{R}^k$ and $\eta : L \rightarrow M$ is an onto bi-Lipschitz transformation. That is there exists constants c_1 and c_2 with $0 < c_1 \leq c_2 < \infty$, such that

$$c_1 d_L(x, y) \leq d_M(\eta(x), \eta(y)) \leq c_2 d_L(x, y)$$

for any $x, y \in L$ where d_L and d_M are the respective metrics on L and M . Then for any $C \subseteq L$, with $|C|_L = 0$, we have $|\eta(C)|_M = 0$ and for any $C' \subseteq L$ with $|L \setminus C'|_L = 0$, $|\eta(L \setminus C')|_M = 0$ where $|\cdot|_L$ (respec $|\cdot|_M$) denotes induced measure on L (respec M).

That is η preserves a *zero – full* law.

In applying Lemma 5, we first need to show that $W_0(m, m-1; \psi) \times \mathbb{I}^{(n-m+1)(m-1)}$ has full Lebesgue measure in $\mathbb{I}^{(m-1)(n+1)}$. Theorem 1 implies that $|W_0(m, m-1; \psi)|_{m(m-1)} = 1$ if $\sum_{r=1}^{\infty} \psi(r)^{m-1} = \infty$ and a straightforward application of Fubini's Theorem gives

$$|W_0(m, m-1; \psi) \times \mathbb{I}^{(n-m+1)(m-1)}|_{(m-1)(n+1)} = 1$$

as the Lebesgue measure of a product of two sets is simply the product of the measures of the two sets. It follows then that $W_0(m, m-1; \psi) \times \mathbb{I}^{(n-m+1)(m-1)}$ is full in $\mathbb{I}^{(m-1)(n+1)}$, as required.

It remains to prove that A , the image of $W_0(m, m-1; \psi) \times \mathbb{I}^{(n-m+1)(m-1)}$ under η is full in Γ . To do this we use local charts on Γ . As Γ is an $(m-1)(n+1)$ -dimension smooth manifold, we know that there is a countable atlas for Γ . Take any chart in the atlas, say (O, ν) where O is an open set in $\mathbb{R}^{(m-1)(n+1)}$ and ν is the (local) diffeomorphism from O to Γ . Now, η is invertible and $\eta^{-1}(\nu(O))$ is in $\mathbb{I}^{(m-1)(n+1)}$. We have just shown that $W_0(m, m-1; \psi) \times \mathbb{I}^{(n-m+1)(m-1)}$ has full measure and so therefore must its intersection with $\eta^{-1}(\nu(O))$. It follows then that η of this intersection must have the same induced measure on Γ as $\nu(O)$ does. We can repeat this argument for each element of the atlas of Γ and it follows that $\eta(A)$ must be full in Γ as required.

This completes the proof of Theorem 2.

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